# Derivation of Gas Loading Equations

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The fundamental relationship of the dissolved gas model is described by a differential equation:

$$\frac{dP}{dt} = k(P_I - P)$$

which states that the instantaneous rate of change of inert gas pressure, dP/dt, in a hypothetical tissue compartment is proportional to a time constant, k, multiplied by the gradient between the inspired inert gas pressure, $P_I$ , and the compartment inert gas pressure, P. This kind of relationship is common in the natural sciences; Newton's Law of Cooling, for example. The main principle is that a gradient is the driving force behind the rate of change in the amount of something.

Our goal is to solve this equation for compartment inert gas pressure as a function of time, P(t). There are two scenarios we consider for diving applications. The first is when the inspired inert gas pressure, $P_I$, remains constant such as during a constant depth profile. The second is when $P_I$ changes with respect to time such as during ascent and descent profiles.

**<u>Derivation of the Haldane Equation</u>**

During a constant depth profile, $P_I$ remains constant and the first-order differential equation can be solved by integration of separable equations. Limits of integration are the initial and final conditions:

$$\frac{dP}{P_I - P} = kdt$$

$$-\int_{P_o}^{P_f=P} \frac{-dP}{P_I - P} = \int_{t_o=0}^{t_f=t} kdt$$

$$-\ln|P_I - P|\Big|_{P_o}^{P_f=P} = kt\Big|_{t_o=0}^{t_f=t}$$

$$-\ln|P_I - P| - (-\ln|P_I - P_O|) = kt - 0$$

$$\ln|P_I - P_O| - \ln|P_I - P| = kt$$

$$\ln\left|\frac{P_I - P_O}{P_I - P}\right| = kt$$

$$\exp\left[\ln\left|\frac{P_I - P_O}{P_I - P}\right|\right] = \exp(kt)$$

$$\frac{P_I - P_O}{P_I - P} = e^{kt}$$

$$\frac{P_I - P_O}{e^{kt}} = P_I - P$$

$$(P_I - P_O)e^{-kt} = P_I - P$$

$$P = P_I - (P_I - P_O)e^{-kt}$$

Now, factor out a $(-1)$ from the quantity in parentheses:

$$P = P_I + (P_O - P_I)e^{-kt}$$

If $P_O$ is added to both sides of the equation, it can be rearranged and expressed in the familiar form of the Haldane Equation:

$$P = P_O + (P_I - P_O)\left(1 - e^{-kt}\right)$$

where:
P or P(t) = compartment inert gas pressure as a function of time
$P_O$ = initial compartment inert gas pressure
$P_I$ = inspired inert gas pressure
e = base of natural logarithms
k = time constant = ln2/half-time
t = time (of exposure or interval)

**Derivation of the Schreiner Equation**

We recognize that the fundamental relationship of the dissolved gas model is a linear first-order ordinary differential equation and the pressures involved are functions of time:

$$\frac{dP}{dt} = k\left[P_I(t) - P(t)\right]$$

As a simplifying assumption in this case, we stipulate that $P_I$ changes at a constant rate corresponding to a constant rate of ascent or descent, e.g. ±10 msw/min. To begin the solution for the differential equation, we must first find an expression for inspired inert gas pressure as a function of time, $P_I(t)$. Based on delivery of the breathing mix by a demand regulator, the inspired inert gas pressure is the fraction of inert gas multiplied by the ambient pressure,

$$P_I = F_{IG} \cdot P_{AMB}$$

In this case, the ambient pressure changes at a constant rate, c,

$$\frac{dP_{AMB}}{dt} = c$$

and the rate of change of inspired inert gas pressure is proportional to the rate of change of ambient pressure:

$$\frac{dP_I}{dt} = F_{IG} \cdot \frac{dP_{AMB}}{dt} = F_{IG} \cdot c$$

To solve for $P_I$ as a function of time, we apply integration of separable equations. Limits of integration are the initial and final conditions:

$$\int_{P_{Io}}^{P_{If}=P_I} dP_I = \int_{t_o=0}^{t_f=t} F_{IG} \cdot c \, dt$$

$$P_I \Big|_{P_{Io}}^{P_{If}=P_I} = F_{IG} \cdot ct \Big|_{t_o=0}^{t_f=t}$$

$$P_I - P_{Io} = F_{IG} \cdot ct - 0$$

$$P_I(t) = P_{Io} + F_{IG} \cdot ct$$

At this point we can consolidate the product of the constants, $F_{IG} \cdot c$, into one constant, R, which is the rate of change of inspired inert gas pressure with respect to time:

$$P_I(t) = P_{Io} + Rt$$

We can now insert this expression back into the differential equation:

$$\frac{dP}{dt} = k\left[P_{Io} + Rt - P(t)\right]$$

The next step is to place this linear first-order differential equation into standard form:

$$\frac{dP}{dt} + kP(t) = kP_{Io} + kRt$$

which is of the form,

$$\frac{dP}{dt} + X(t)P(t) = Y(t)$$

We need to determine an integrating factor, $\mu(t)$, to be multiplied on both sides of the equation such that the left-hand side of the multiplied equation,

$$\mu(t)\frac{dP}{dt} + \mu(t)X(t)P(t)$$

is just the derivative of the product, $\mu(t)P(t)$.

In this case, X(t) = k and the integrating factor is given by:

$$\mu(t) = \exp\left(\int k dt\right) = e^{kt}$$

We now multiply both sides of the equation by $e^{kt}$:

$$\frac{dP}{dt}e^{kt} + kP(t)e^{kt} = kP_{Io}e^{kt} + kRte^{kt}$$

Recognizing that the left-hand side of this equation is a result of the product rule,

$$\frac{d}{dt}(UV) = U\frac{dV}{dt} + V\frac{dU}{dt}$$

it can be expressed as:

$$\frac{d}{dt}\left[P(t)e^{kt}\right] = kP_{Io}e^{kt} + kRte^{kt}$$

We can now integrate both sides of the equation with respect to time keeping in mind that the solution for a first-order differential equation will evoke a single arbitrary constant, $C_1$:

$$\int \frac{d}{dt}\left[P(t)e^{kt}\right]dt = \int kP_{Io}e^{kt}dt + \int kRte^{kt}dt + C_1$$

$$P(t)e^{kt} = P_{Io}e^{kt} + kR\int te^{kt}dt + C_1$$

The remaining integral is evaluated using integration by parts, $\int UdV = UV - \int VdU$,

where $U = t,\ dU = 1,\ dV = e^{kt},$ and $V = \frac{1}{k}e^{kt}$:

$$P(t)e^{kt} = P_{Io}e^{kt} + kR\left(\frac{t}{k}e^{kt} - \frac{1}{k^2}e^{kt}\right) + C_1$$

$$P(t)e^{kt} = P_{Io}e^{kt} + Re^{kt}\left(t - \frac{1}{k}\right) + C_1$$

Next, divide both sides of the equation by $e^{kt}$ in order to solve for P(t):

$$P(t) = P_{Io} + R\left(t - \frac{1}{k}\right) + \frac{C_1}{e^{kt}}$$

$$P(t) = P_{Io} + R(t - 1/k) + C_1 e^{-kt}$$

We can solve for the constant of integration, $C_1$, by recognizing that in the case of a first-order equation, the initial conditions reduce to the single requirement,

$$P(t_o) = P_O$$

So, with $t_o = 0$ this gives:

$$P_O = P_{Io} + R(0 - 1/k) + C_1 e^0$$

and

$$C_1 = P_O - P_{Io} + R/k$$

The unique solution is thus,

$$P(t) = P_{Io} + R(t - 1/k) + (P_O - P_{Io} + R/k)e^{-kt}$$

or, by factoring a $(-1)$ out of the last set of parentheses:

$$P(t) = P_{Io} + R(t - 1/k) - (P_{Io} - P_O - R/k)e^{-kt}$$

This latter form is the Schreiner Equation as originally published*
where
P(t) = compartment inert gas pressure as a function of time
$P_O$ = initial compartment inert gas pressure
$P_{Io}$ = initial inspired inert gas pressure
R = rate of change of inspired inert gas pressure with change in ambient pressure
e = base of natural logarithms
k = time constant = ln2/half-time
t = time (of exposure or interval)

Note that when R = 0 in the Schreiner Equation, it reduces to the Haldane Equation used for constant depth applications:

$$P(t) = P_O + (P_I - P_O)\left(1 - e^{-kt}\right)$$

* Schreiner, H.R. and Kelley, P.L. 1971. A pragmatic view of decompression. In: Lambertsen, C.J., ed. Underwater physiology: Proceedings of the fourth symposium on underwater physiology. Academic Press, New York, 205-219.